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Trailing edge flows

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The flow is examined in the neighbourhood of the trailing edge of slender aerodynamic shapes which terminate in either a cusp or a wedge. The manner in which the boundary layer reacts to the rapidly varying pressure field in such regions is analyzed using the method of matched asymptotic expansions. The case of a wedge is examined in greater detail and a criterion for separation to occur is established.

1. Introduction

In this paper we are concerned with the response of a steady laminar boundary layer to the rapid variation of pressure which inviscid potential theory predicts to occur near the trailing edge of certain slender non-lifting aerodynamic shapes. The investigation is part of a programme of study of the structure of the solutions of the Navier–Stokes equations in the neighbourhood of the trailing edges of aerodynamic shapes when the Reynolds number R of the flow is very large. Previous papers (Stewartson 1968, 1969; Messiter 1969) have been concerned with flat plates at zero incidence and here we consider bodies with thickness. At a later stage it is hoped to combine the earlier work with this study in order to deepen our knowledge of trailing edge flows for aerodynamically well designed shapes. For the present, however, we confine our attention to boundary-layer theory and we establish a criterion for separation to occur and make an estimate of the distance from the trailing edge at which separation takes place for a class of shapes terminating in a cusp or a small non-zero angle.

The study is believed to be useful for three reasons. First the characteristics of a boundary layer, and particularly the point of separation, when subjected to a rapidly increasing pressure, have not been fully analyzed. Earlier rational studies, for example, Stewartson (1951) and Lighthill (1953) have restricted themselves to linear equations, while the approximate method developed by Stratford (1954) and Gadd (1957) has not been mathematically secured particularly with respect to separation. The idea behind Stratford's method, which has been successfully applied to a variety of boundary layer flows, is that the slow moving, innermost part of the boundary layer reacts more readily to a rapid variation in pressure than the outer part. Throughout the region in which significant changes take place in the inner part, the flow in the outer part is assumed to behave as an inviscid flow. Approximate solutions in the two parts are patched heuristically. In this paper we employ the method of matched asymptotic expansions to the trailing edge flows in question and show how Stratford's method can be set on a formal rational basis, which can be regarded as the natural extension of the linear theories of Stewartson (1951) and Lighthill. We find that the essential assumptions made by Stratford are valid but not all the details of the flow are accurately reproduced by the approximate method.

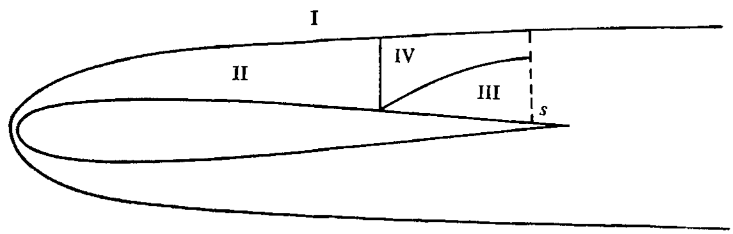

FIGURE 1

Of the trailing edge flows considered, prime attention is given to the case when the body shape near the trailing edge takes the form of a symmetrically disposed wedge of small non-zero angle $2\pi\alpha$. In this case the trailing edge is a stagnation point of the inviscid irrotational flow field and on the basis of an inviscid flow plus a laminar boundary layer the flow field up to the trailing edge can be divided into four regions as indicated in figure 1 (which is not drawn to scale): (*a*) region I in which the flow is irrotational; (*b*) region II, the classical Prandtl boundary layer of thickness $O(R^{-\frac{1}{2}})$ between I and the body. Now we find that near the trailing edge, and specifically within a distance $O(\alpha^{\frac{3}{2}})$ of it, region II splits into (*c*) region III, an inner boundary layer of thickness $O(\alpha^{\frac{1}{2}}R^{-\frac{1}{2}})$ embedded in (*d*) region IV. This is an essentially inviscid region of thickness $O(R^{-\frac{1}{2}})$ into which vorticity from region II is convected. The structure of the solution in these four regions is analyzed in detail using perturbation techniques. The response of the flow in III to the rapidly varying pressure leads to separation taking place before the trailing is reached and within a distance $O(\alpha^{\frac{3}{2}})$ of it. Thus the structure of the boundary layer on the basis assumed has been elucidated which is the second reason for the present study.

However, we note that according to the theory of separation, with a prescribed pressure gradient, it is accompanied in general by a singularity and there seems no reason to doubt its existence here. Such a singularity is also to be expected from the experimental evidence of incompressible flows, for separation then invariably leads to a break-away of the mainstream from the wall, for which no mathematical explanation is forthcoming if the boundary layer hypotheses hold downstream of separation. This phenomenon reveals a weakness in the present theory since the assumed potential flow outside the boundary layer is incorrect downstream of separation and must be distorted to a greater or lesser degree upstream. We are inclined to believe that if $\alpha \ll 1$ this distortion is not disastrous in the regions examined in this paper and that its value as a first step towards unravelling the flow structure near the trailing edge of a wedge is not thereby greatly diminished.

This brings us to the third reason for the study, which is to give a qualitative criterion for the inhibition of separation. For a flat plate the change in character of the boundary layer at the trailing edge leads to a dramatic fall in the displacement thickness which in turn induces a weak but favourable pressure gradient just upstream. It is to be expected that replacing the plate by a wedge will not greatly affect this result so that the possibility arises of the retarding effect of the inviscid stagnation point being balanced by the induced accelerating effect of the boundary layer leading to the inhibition of separation. The present study is then useful for setting up the appropriate upstream boundary conditions for the region where these two pressure gradients balance. The study of this new region is complicated and is deferred to a later paper but a few comments are in order. First a simple argument is set out at the end of this paper to show that separation does not occur if $\alpha \ll R^{-\frac{1}{4}}$. Second it is noted that in the case of the flat plate the favourable pressure gradient is set up within a distance $O(R^{-\frac{3}{8}})$ of the trailing edge so that the arguments of this paper are certainly incomplete if

$$\alpha^{\frac{3}{2}} = O(R^{-\frac{3}{8}}),$$

or

$$\alpha = O(R^{-\frac{1}{4}}), \tag{1.1}$$

and it is believed that this is the criterion for the inhibition of separation.

2. Trailing edge flows

We consider high Reynolds number, irrotational steady flow over a slender aerodynamic shape which terminates in either a cusp or a wedge as shown in figure 2.

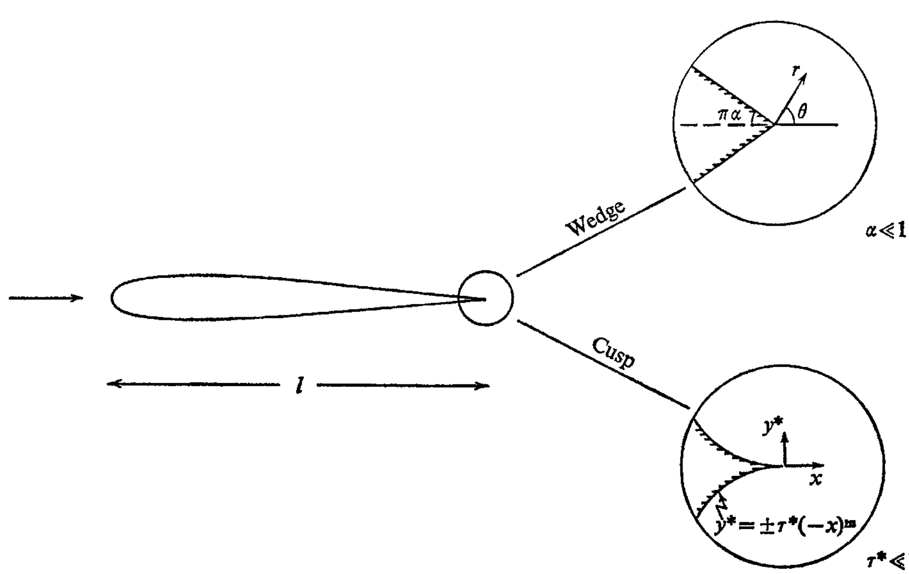

FIGURE 2

For $R \gg 1$ where $R = U_\infty l/\nu$ we initially neglect the viscous terms in the Navier–Stokes equations and first consider the inviscid, irrotational flow field which is made determinate by applying the Kutta–Joukowski condition at the trailing edge. Although in principle the complete irrotational flow field can be computed we shall content ourselves with local solutions valid in the neighbourhood of the trailing edge.

In a suitable dimensionless form, with l as reference length, we have for the wedge the well-known result

$$\psi' \sim r^{1+\alpha}\sin\theta \quad (\alpha \ll 1), \tag{2.1}$$

whilst for the cusp Van Dyke (private communication) has observed that

$$\psi' \sim \left(r\sin\theta - \tau^* r^m \frac{\sin m\theta}{\sin m\pi}\right) \quad (\tau^* \ll 1), \tag{2.2}$$

where ψ' is the two-dimensional stream function. These solutions have to be corrected for the slip they predict at the solid surface by introducing Prandtl's concept of a boundary layer (which has thickness $O(R^{-\frac{1}{2}})$ and within which $\psi' = O(R^{-\frac{1}{2}})$).

We now turn to the boundary layer associated with the inviscid flow past slender aerodynamic shapes with particular emphasis on the neighbourhood of the trailing edge. Using (2.1) the inviscid slip velocity near the trailing edge of a wedge $\propto s^\alpha$ where s measures distance from the trailing edge. If the trailing edge is a cusp (2.2) indicates that the corresponding slip velocity $\propto 1+\tau' s^{m-1}$ where $|\tau'| \ll 1$. Further $\tau' > 0$ if $1 < m < \frac{3}{2}$ so that the pressure gradient is adverse near $s = 0$, $\tau' = 0$ if $m = \frac{3}{2}$ and the aerofoil has a Joukowsky form, while $\tau' < 0$ if $m > \frac{3}{2}$ so that the pressure gradient is favourable near $s = 0$. In order to determine the principal properties of the boundary layer near $s = 0$ the following simplified problem will be studied. Consider a flat plate of unit length in the (x, y) plane where $yR^{-\frac{1}{2}} = y^*$ measures distance normal to the plate, x distance downstream from the trailing edge and $\xi = 1+x$ distance downstream from the leading edge. If $(u, vR^{-\frac{1}{2}})$ are the velocity components in the (x, y) directions the boundary layer equations are then

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U_1\frac{dU_1}{dx} + \frac{\partial^2 u}{\partial y^2}, \tag{2.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.4}$$

with boundary conditions

$$\left.\begin{aligned} &u = v = 0 \quad \text{at} \quad y = 0, \quad -1 < x < 0;\\ &u = U_1(-1) \quad \text{if} \quad x = -1, \quad y > 0;\\ &u \to U_1(x) \quad \text{as} \quad y \to \infty. \end{aligned}\right\} \tag{2.5}$$

Here
$$U_1(x) = (-x)^\alpha \quad \text{or} \quad 1+\frac{\alpha}{m-1}(-x)^{m-1}, \tag{2.6}$$

where $0 < \alpha \ll 1$.

As posed this problem differs from that for thin wedge and cusped aerofoils only in the definition of $U_1(x)$ and even there through terms which are significant only near the leading edge or which are smooth and uniformly of order α in $-1 < x < 0$. They will not affect the *character* of the boundary layer solution in the neighbourhood of the trailing edge.

Thus the inviscid irrotational flow past slender aerodynamic shapes, some of whose properties are given in (2.1), (2.2) and valid in region I of figure 1, induces over the majority of the aerofoil a boundary layer whose properties, excluding the leading edge region, differ only slightly from that for uniform flow past a flat plate. We shall refer to that part of the boundary layer as region II and note that since $dU_1/dx \to \infty$ as $x \to 0-$ it must come to an end when $|x| \ll 1$ in some sense. In order to investigate the nature of its breakdown we introduce the scaling

$$x = \Delta X, \tag{2.7}$$

where $\Delta \ll 1$ is a constant to be found and which is explicitly obtained in (2.12) below. However, we make the preliminary assumption, found to be consistent *a posteriori* that $\Delta \gg e^{-1/\alpha}$ for all m of interest here which enables us to replace $U_1 dU_1/dx$ by $-\alpha/(-x)^{2-m}$ in (2.3). Using (2.7) it can readily be shown that the pressure and viscous terms in (2.3) are respectively $O(\alpha\Delta^{m-1})$, $O(\Delta)$ compared to the inertia terms so that to first order the flow in this region is effectively inviscid. We shall refer to this inviscid region as region IV and we observe that matching the solution in this region with the Blasius solution $u_B = f'_B(y/\xi^{\frac{1}{2}})$ gives $u = u_B = f'_B(y)$ to first order in α in IV. However, although the pressure forces do not modify the velocity profile, to first order, in region IV it can be shown that close to the trailing edge there is a thin inner boundary layer in which the pressure and viscous terms in (2.3) are comparable with the inertia terms. To investigate this inner boundary layer, which we shall designate region III we set, along with (2.7)

$$y = \delta Y, \tag{2.8}$$

where $\delta \ll 1$ (see (2.12) below). To find the scaling for u and v appropriate to this inner boundary layer region III we observe that the inner expansion of the solution in IV, written in terms of the variables of region III, yields

$$\partial u_B/\partial Y = O(\lambda\delta) \quad \text{where} \quad \lambda = f''_B(0) = 0{\cdot}3321.$$

This determines the scale for u, that for v follows from continuity. Writing

$$u = \lambda\delta U(X, Y), \quad v = \lambda\delta^2 V(X, Y)/\Delta, \tag{2.9}$$

where $U = O(1)$, $V = O(1)$ and substituting in (2.3) we obtain

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \frac{-\alpha}{\lambda^2\Delta^{1-m}\delta^2(-X)^{2-m}} + \frac{\Delta}{\lambda\delta^3}\frac{\partial^2 U}{\partial Y^2}, \tag{2.10}$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0. \tag{2.11}$$

Consequently the inertia, pressure and viscous terms in (2.10) will be comparable if we choose

$$\delta = (\alpha\lambda^{m-3})^{1/(5-3m)}, \quad \Delta = (\alpha\lambda^{-\frac{4}{3}})^{3/(5-3m)} \tag{2.12}$$

and the equations governing the flow in this inner boundary layer region III are

$$\left.\begin{aligned} U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y} &= -\frac{1}{(-X)^{2-m}}+\frac{\partial^2 U}{\partial Y^2}, \\ \frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y} &= 0. \end{aligned}\right\} \tag{2.13}$$

In addition to the boundary conditions $U = V = 0$ at $Y = 0$ the matching requirements and (2.13) show that

$$\left.\begin{aligned} U - Y &\to g(X), \\ V - g'(X)\,Y &\to h(X), \end{aligned}\right\} \quad \text{as} \quad Y \to \infty \tag{2.14}$$

where $h + gg' = -(-X)^{m-2}$, together with

$$U \to Y \quad \text{as} \quad X \to -\infty \quad \text{for all } Y. \tag{2.15}$$

We note that the arguments associated with the discussion of the inner boundary layer leading to (2.13) will be valid as long as the pressure gradient due to the irrotational flow is large compared with that due to the change in character of the boundary layer at the trailing edge. The results of Goldburg & Cheng (1961) for the flow at the trailing edge of a flat plate indicate that we may expect this pressure gradient to be favourable and to be $O(R^{-\frac{1}{2}}/(-x)^{\frac{5}{3}})$. Thus the arguments set out above will only be valid when $\alpha(-x)^{m-2} \gg R^{-\frac{1}{2}}(-x)^{-\frac{5}{3}}$ which, using (2.12) for the region in question, implies

$$\alpha R^{\frac{1}{8}(5-3m)} \gg 1, \tag{2.16}$$

which is consistent with (1.1) when $m = 1$.

In the next section we discuss the structure of the flow in our various regions in more detail for the case $m = 1$ when the trailing edge geometry is that of a wedge. Before leaving this section, however, we observe that equations (2.13) imply, since the inner boundary layer now suffers an adverse pressure gradient $O(1)$, that separation will take place at a finite value of X and thus on the original scale at a distance $O[\alpha^{3/(5-3m)}]$ from the tip.

3. Solution structure for $m = 1$ (wedge)

With the solution in region I given by (2.1) we now consider the solution in each of the regions II–IV discussed in § 2 and show how these solutions are matched taking $m = 1$ and choosing $\alpha \ll 1$ as the perturbation parameter.

Region II

In this classical boundary layer region, dominated by the Blasius flow, we note that the pressure term in the boundary layer equations may be written as

$$U_1\frac{dU_1}{dx} = \frac{\alpha}{x}\{1 + 2\alpha\log(-x) + O(\alpha^2)\}. \tag{3.1}$$

If, in our perturbation solution in II, we retain only the first term in (3.1) then our governing equations may be written as

$$\left.\begin{aligned} u\frac{\partial u}{\partial \xi}+v\frac{\partial u}{\partial y} &= -\alpha\sum_{n=0}^{\infty}\xi^n+\frac{\partial^2 u}{\partial y^2},\\ \frac{\partial u}{\partial \xi}+\frac{\partial v}{\partial y} &= 0,\end{aligned}\right\} \tag{3.2}$$

with

$$\left.\begin{aligned} u = v &= 0 \quad \text{on} \quad y = 0,\\ u &\sim 1-\alpha\sum_{n=1}^{\infty}\xi^n/n \quad \text{as} \quad y\to\infty.\end{aligned}\right\} \tag{3.3}$$

Making a formal perturbation to the Blasius solution in this region we write the stream function ψ as

$$\psi = \xi^{\frac{1}{2}}f_B(\tilde{\eta})+\alpha\sum_{n=0}^{\infty}\xi^{n+\frac{3}{2}}f_n(\tilde{\eta})+O(\alpha^2), \tag{3.4}$$

where $f_B(\tilde{\eta})$ is the Blasius function and $\tilde{\eta} = y/\xi^{\frac{1}{2}}$. Substituting (3.4) into (3.2) and equating coefficients of powers of ξ shows that $f_n(\tilde{\eta})$ satisfies

$$\left.\begin{aligned} f_n'''+\tfrac{1}{2}f_Bf_n''-(n+1)f_B'f_n'+(n+\tfrac{3}{2})f_B''f_n &= 1,\\ f_n(0) = f_n'(0) = 0, \quad f_n'(\infty) &= -(n+1)^{-1}.\end{aligned}\right\} \tag{3.5}$$

Although solutions of (3.5) cannot be written in closed form for arbitrary n solutions for large n will yield valuable information, since they will lead to the terms that are singular as $\xi\to 1$.

For $n \gg 1$ we write

$$(n+1)f_n = \Phi_n(\tilde{\eta})+o(1), \tag{3.6}$$

so that

$$\left.\begin{aligned} f_B'\Phi_n'-f_B''\Phi_n+1 &= 0,\\ \Phi_n' \to -1 \quad &\text{as} \quad \tilde{\eta}\to\infty.\end{aligned}\right\} \tag{3.7}$$

We have ignored, for the moment, the boundary conditions at $\tilde{\eta} = 0$ since the solution of (3.7), which neglects the highest derivatives in (3.5), will represent an outer solution valid for any fixed $\tilde{\eta} > 0$ as $n\to\infty$. The solution is

$$\begin{aligned} \Phi_n &= f_B'(\tilde{\eta})\int_{\tilde{\eta}}^{\infty}\left\{\frac{1}{f_B'^2(t)}-1\right\}dt-\tilde{\eta}f_B'(\tilde{\eta})+C_nf_B'(\tilde{\eta})\\ &= \Lambda_1(\tilde{\eta})+C_nf_B'(\tilde{\eta}), \end{aligned} \tag{3.8}$$

where C_n is a constant to be found. From (3.6) and (3.8) we see that as $\tilde{\eta}\to 0$

$$\Phi_n \to 1/\lambda+C_n\lambda\tilde{\eta}, \tag{3.9}$$

and for no choice of the constant C_n can both boundary conditions at the wall be satisfied. To discuss the inner region we write

$$\left.\begin{aligned} \tilde{\eta} &= n^{-\frac{1}{2}}\zeta,\\ nf_n &= F_n(\zeta)+o(1),\end{aligned}\right\} \tag{3.10}$$

where the choice of variables in (3.10) is governed by the fact that the viscous term in (3.5) must, in the inner region, be comparable with the other terms and that this inner solution must match with (3.8). The equation satisfied by $F_n(\zeta)$ is, from (3.5) and (3.10) for $n \geqslant 1$

$$F_n''' - \lambda\zeta F_n' + \lambda F_n = 1, \tag{3.11}$$

with

$$F_n(0) = F_n'(0) = 0. \tag{3.12}$$

In order to achieve a match with the outer solution we must exclude the complementary function of (3.11) which is exponentially large when $\zeta \gg 1$ and we deduce that

$$\left.\begin{aligned} F_n'' &= a_n \mathrm{Ai}\,(\lambda^{\frac{1}{3}}\zeta), \\ \text{where}\qquad a_n &= \lambda^{-\frac{1}{3}}/\mathrm{Ai}'(0) = -\lambda^{-\frac{1}{3}}3^{\frac{1}{3}}(\tfrac{1}{3})! \end{aligned}\right\} \tag{3.13}$$

and Ai is the Airy function. Thus from (3.12) and (3.13) we have

$$F_n - a_n\lambda^{-\frac{1}{3}}\zeta\int_0^\infty \mathrm{Ai}(t)\,dt \to \lambda^{-1} \quad \text{as} \quad \zeta \to \infty, \tag{3.14}$$

and matching the solution with (3.9)

$$n^{-\frac{1}{3}}C_n \to -\lambda^{-\frac{2}{3}}3^{\frac{1}{3}}(\tfrac{1}{3})! = C \quad \text{as} \quad n \to \infty. \tag{3.15}$$

If we now denote the coefficient of α in (3.4), the perturbation to the Blasius solution, by ψ_P then we have shown that for $\tilde{\eta} = O(1)$, ψ_P may be written as

$$\psi_P = -\xi^{\frac{1}{2}}\Lambda_1(\tilde{\eta})\log(1-\xi) + C\xi^{\frac{1}{2}}f_B'(\tilde{\eta})\sum_{n=0}^{\infty}\frac{\xi^{n+1}}{(n+1)^{\frac{2}{3}}} + \Lambda_2(\xi,\tilde{\eta}), \tag{3.16}$$

where the function $\Lambda_2(\xi,\tilde{\eta})$ has not been determined but, unlike the other two terms, remains finite as $\xi \to 1$.

One of the conditions to be imposed upon the solution in region IV is that it must match asymptotically as $\alpha \to 0$ with the solution in region II. Anticipating the matching condition we see, from (3.4) and (3.16) that when $1-\xi \ll 1$ (with $-X \gg 1$) and $\tilde{\eta} > 0$

$$\psi = f_B(y) - \alpha\Lambda_1(y)\log(-x) + \alpha\bar{C}f_B'(y)(-x)^{-\frac{1}{3}} + \alpha\Lambda_2(y) + O(\alpha^2), \tag{3.17}$$

where $\bar{C} = -2\pi(\frac{1}{3})!/\lambda^{\frac{2}{3}}3^{\frac{1}{6}}(-\frac{1}{3})!$ and $\xi = 1+x$. Similarly the wall shear stress as calculated in region III must match asymptotically with that in region II. We note that from (3.10) and (3.13)

$$\left(\frac{\partial^2\psi}{\partial y^2}\right)_{y=0} = \frac{\lambda}{\xi^{\frac{1}{2}}} - \alpha\left(\frac{(-\frac{2}{3})!}{\lambda^{\frac{1}{3}}3^{\frac{1}{3}}(-\frac{1}{3})!}\right)\sum_{n=1}^{\infty}\frac{\xi^{n+\frac{1}{2}}}{n^{\frac{1}{3}}} + \alpha\xi^{-1}\bar{\Lambda}_2''(\xi,0) + O(\alpha^2), \tag{3.18}$$

where $\bar{\Lambda}_2$, an inner solution corresponding to Λ_2, remains finite as $\xi \to 1$ so that when $1-\xi \ll 1$ (with $-X \gg 1$)

$$\left(\frac{\partial^2\psi}{\partial y^2}\right)_{y=0} \approx \lambda - \alpha\left(\frac{2\pi}{\lambda^{\frac{1}{3}}3^{\frac{5}{6}}(-\frac{1}{3})!}\right)(-x)^{-\frac{2}{3}} + \alpha\bar{\Lambda}_2''(1,0). \tag{3.19}$$

The result (3.19) indicates that we can expect a region close to the wall, at a distance $O(\alpha^{\frac{3}{4}})$ from the tip, in which the solution cannot be represented by a perturbation to the Blasius solution. This is in accord with our findings in § 2.

At this stage we remark again that the example being studied is for illustrative purposes only. In flows of practical interest past slender aerodynamic shapes the true slip velocity is expected to differ from (2.6) by terms of relative order α which will, however, remain smooth as $x \to 0-$. These terms modify (3.16) and (3.18) only through Λ_2 and $\bar{\Lambda}_2''$ and do not affect the principal features of the solution in regions III and IV which we now discuss.

Region III

For the case of a wedge we see from (2.7), (2.8), (2.9) and (2.12) that the variables appropriate to region III are given by

$$\left.\begin{aligned} x &= \lambda^{-2}\alpha^{\frac{3}{4}}X, \quad & y &= \lambda^{-1}\alpha^{\frac{1}{2}}Y, \\ u &= \alpha^{\frac{1}{4}}U, & v &= \lambda\alpha^{-\frac{1}{4}}V. \end{aligned}\right\} \tag{3.20}$$

On setting $\alpha = 0$ the governing equations for U, V are given by (2.13) with $m = 1$ and the boundary conditions by (2.14) and (2.15).

Before discussing the solution of (2.13) we note that given U the next term in the expansion of u about $\alpha = 0$ can be readily determined. If we retain the first two terms in the pressure gradient for this region, derived from (3.1), the only modification to (2.13) is that the coefficient of X^{-1} is replaced by $1 + 3\alpha\log\alpha$ whilst (2.14) and (2.15) are unaltered. Thus the expression $u = \alpha^{\frac{1}{4}}U$ in (3.20) is replaced by

$$u = \alpha^{\frac{1}{4}}\left[U + \tfrac{3}{2}\alpha\log\alpha\left\{U - 3X\frac{\partial U}{\partial X} - Y\frac{\partial U}{\partial Y}\right\} + O(\alpha)\right]. \tag{3.21}$$

We turn now to a discussion of the properties of $U(X, Y)$ and in particular the asymptotic forms which are used to effect a match with regions II and IV. As $X \to -\infty$ the solution of (2.13) can be written in the form

$$\Psi = \sum_{m=0}^{\infty} (-X)^{\frac{2}{3}-\frac{2}{3}m}\bar{f}_m(\eta), \tag{3.22}$$

where
$$U = \frac{\partial\Psi}{\partial Y}, \quad V = -\frac{\partial\Psi}{\partial X} \quad \text{and} \quad \eta = Y/(-X)^{\frac{1}{3}}. \tag{3.23}$$

Of the functions $\bar{f}_m(\eta)$, $\bar{f}_0 = \frac{1}{2}\eta^2$ and for $m \geqslant 1$

$$\left.\begin{aligned} &3\bar{f}_1''' - \eta^2\bar{f}_1'' = 3, \\ &3\bar{f}_2''' - \eta^2\bar{f}_2'' - 2\eta\bar{f}_2' + 2\bar{f}_2 = \bar{f}_1'^2, \\ &3\bar{f}_3''' - \eta^2\bar{f}_3'' - 4\eta\bar{f}_3' + 4\bar{f}_3 = 4\bar{f}_1'\bar{f}_2' - 2\bar{f}_1''\bar{f}_2, \text{ etc.} \end{aligned}\right\} \tag{3.24}$$

together with
$$\bar{f}_m(0) = \bar{f}_m'(0) = 0, \quad \bar{f}_m'(\infty) \text{ bounded.} \tag{3.25}$$

We note then that $g(X)$, in (2.14), is given as $X \to -\infty$ by

$$g(X) = -\sum_{m=1}^{\infty} (-X)^{\frac{1}{3}-\frac{2}{3}m}\bar{f}_m'(\infty). \tag{3.26}$$

The linear equations (3.26) can be solved successively without any great difficulty bearing in mind that each has a complementary function f_c for which, as $\eta \to \infty$,

$$f_c \sim \eta^{-4} \exp(\eta^3/9). \tag{3.27}$$

From this asymptotic solution for Ψ we can show that when $-X \gg 1$, $\eta \gg 1$

$$\frac{\partial U}{\partial Y} \sim 1 - \frac{3}{Y^2} + \frac{6\bar{C}_1(-X)^{-\frac{1}{3}}}{Y^3} + \frac{18}{Y^4}\{\log Y - \tfrac{1}{3}\log(-X)\} + \frac{\bar{C}_2}{Y^4} + \frac{\bar{C}_3(-X)^{-1}}{Y^3} - 9\bar{C}_1^2 \frac{(-X)^{-\frac{2}{3}}}{Y^4} + \dots, \tag{3.28}$$

where the constants $\bar{C}_1$, $\bar{C}_2$ and $\bar{C}_3$ are known.

We next consider the solution of (2.13) for finite X as $Y \to \infty$. Thus we write

$$\left.\begin{aligned} U &= Y - g(X) + \tilde{u}, \\ V &= Yg'(X) + h(X) + \tilde{v}, \end{aligned}\right\} \tag{3.29}$$

where $|\tilde{u}|$, $|\tilde{v}| \ll 1$ and substitute in (2.13). Neglecting products of small terms we have, finally, as an equation for $\tilde{u}_Y = \partial\tilde{u}/\partial Y$,

$$\tilde{u}_{YYY} - (g'Y + h)\tilde{u}_{YY} + (g - Y)\tilde{u}_{XY} = 0. \tag{3.30}$$

Substituting for h and writing $\phi = u_Y$ we write (3.30) as

$$\phi_X = Y^{-1}\phi_{YY} - (XY)^{-1}\phi_Y + gg'Y^{-1}\phi_Y - g'\phi_Y + gY^{-1}\phi_X, \tag{3.31}$$

and we assume that the terms on the right-hand side are small compared with the left-hand side for large Y. Thus we write

$$\phi = \phi_0 + \phi_1 + \phi_2 + \dots, \tag{3.32}$$

where $\phi_{0X} = 0$. Comparison with (3.28) indicates that

$$\phi_0 = -3/Y^2. \tag{3.33}$$

It is clear that $\phi_1 = O(Y^{-3})$ and thus

$$\phi_{1X} = -\frac{6g'}{Y^3},$$

which gives

$$\phi_1 = -\frac{6g}{Y^3} + \frac{A_1}{Y^3} \tag{3.34}$$

and comparison with (3.28) indicates that $A_1 \equiv 0$. Similarly

$$\phi_{2X} = -\frac{6}{XY^4} - \frac{18gg'}{Y^4}, \tag{3.35}$$

and, using (3.28), we write the solution of (3.35) as

$$\phi_2 = -\frac{6}{Y^4}\log(-X) - \frac{9g^2}{Y^4} + \frac{18\log Y}{Y^4} + \frac{\bar{C}_2}{Y^4}. \tag{3.36}$$

We do not continue the solution beyond this stage. Thus we have, as $Y \to \infty$,

$$\phi = -\frac{3}{Y^2} - \frac{6g(X)}{Y^3} + \frac{18 \log Y}{Y^4} - \frac{[6 \log(-X) + 9g^2 - \bar{C}_2]}{Y^4} + O(Y^{-5}), \quad (3.37)$$

which, as $X \to -\infty$, yields (3.28). It may be verified that the terms neglected in (3.30) make no contribution to (3.37). It is perhaps also worth pointing out that arguments similar in nature to the above show that there is a possible exponential behaviour

$$\frac{\partial^2 U}{\partial Y^2} \sim \frac{X}{Y} \exp\left[-\frac{Y^3}{9X}\right], \quad (3.38)$$

which is in accord with (3.27) and which has been suppressed.

To summarize we have, as $Y \to \infty$,

$$\Psi = \tfrac{1}{2}Y^2 - gY + 3\log Y + \tfrac{1}{2}g^2 - \log(-X) - 3g/Y$$
$$-\frac{1}{2}\frac{\log Y}{Y^2} - \left[\frac{6\log(-X) + 18g^2 + 5 - 2\bar{C}_2}{12Y^2}\right] + O(Y^{-3}). \quad (3.39)$$

Before leaving region III we note from (3.22) that as $X \to -\infty$

$$\left(\frac{\partial^2 \Psi}{\partial Y^2}\right)_{Y=0} = 1 + (-X)^{-\frac{2}{3}} f_1''(0) + O[(-X)^{-\frac{4}{3}}], \quad (3.40)$$

and since from (3.24)

$$f_1''(0) = -\int_0^\infty e^{-\frac{1}{9}\eta^3}\, d\eta = -2\pi/3^{\frac{5}{6}}(-\tfrac{1}{3})!$$

we see with the help of (3.20) that (3.19) and (3.40) match as required.

We now consider the solution in the effectively inviscid region within which the inner boundary layer III is embedded.

Region IV

We have already seen in § 2 that to first order region IV is one of inviscid rotational flow. The scales appropriate to region IV may be inferred from § 2 and we write

$$x = \lambda^{-2}\alpha^{\frac{3}{2}}\bar{X}, \quad y = \bar{Y} \quad \text{and} \quad \psi = \bar{\psi} \quad (3.41)$$

as the variables for this region where we have taken $\bar{\psi} = O(1)$ on the Prandtl boundary layer scale to effect a match with region II. With the continuity equation satisfied we have, substituting (3.41) into (2.13),

$$\bar{\psi}_{\bar{Y}}\bar{\psi}_{\bar{X}\bar{Y}} - \bar{\psi}_{\bar{X}}\bar{\psi}_{\bar{Y}\bar{Y}} = \frac{\alpha}{\bar{X}} + \lambda^{-2}\alpha^{\frac{3}{2}}\bar{\psi}_{\bar{Y}\bar{Y}\bar{Y}}. \quad (3.42)$$

The boundary conditions are formulated by requiring that the solution match with solutions in adjacent regions. The contributions which regions I, II and III make to the asymptotic forms of $\bar{\psi}$ are obtained by writing (2.1), (3.4) and Ψ of III in terms of the variables (3.41) and formally letting $\alpha \to 0$. Thus with the help of (3.17) and (3.39) we see that

$$\bar{\psi} = \tfrac{1}{2}\lambda\bar{Y}^2 - \alpha^{\frac{1}{2}}\bar{Y}g(\bar{X}) - \frac{3\alpha}{2\lambda}\log\alpha + \frac{\alpha}{\lambda}\{\tfrac{1}{2}g^2 - \log(-\bar{X}) + 3\log\lambda\bar{Y}\} + O[\alpha^{\frac{3}{2}}\log\alpha],$$

as $\bar{Y} \to 0$, (3.43)

$$\bar{\psi} = \bar{Y} + \tfrac{3}{2}(\alpha\log\alpha)\,\bar{Y} + \alpha\bar{Y}\log(-\bar{X}) - 2\alpha\bar{Y}\log\lambda + \Phi(\bar{X},\alpha) + O(\alpha^2), \quad (3.44)$$

as $\bar{Y} \to \infty$, where $\Phi(\bar{X},\alpha)$ arises from solutions of higher order than (2.1) in region I, and

$$\bar{\psi} = f_B(\bar{Y}) + \alpha^{\frac{1}{2}}\lambda^{\frac{3}{2}}\bar{C}f'_B(\bar{Y})\,(-\bar{X})^{-\frac{1}{3}} - \tfrac{3}{2}(\alpha\log\alpha)\,\Lambda_1(\bar{Y})$$
$$-\alpha\{\Lambda_1(\bar{Y})\log(-\bar{X}) + \Lambda_3(\bar{Y})\} + O(\alpha^2), \quad (3.45)$$

as $\bar{X} \to -\infty$, where $\Lambda_3(\bar{Y})$ is not determined.

We seek a solution

$$\bar{\psi} \sim \bar{\psi}_0 + \alpha^{\frac{1}{2}}\bar{\psi}_1 + (\alpha\log\alpha)\,\bar{\psi}_2 + \alpha\bar{\psi}_3 + \ldots. \quad (3.46)$$

Substitute in (3.42) and equate the coefficients of powers of α.

$O(1)$

$$\frac{\partial\bar{\psi}_0}{\partial\bar{Y}}\frac{\partial^2\bar{\psi}_0}{\partial\bar{X}\,\partial\bar{Y}} - \frac{\partial^2\bar{\psi}_0}{\partial\bar{Y}^2}\frac{\partial\bar{\psi}_0}{\partial\bar{X}} = 0. \quad (3.47)$$

Matching the inviscid solution $\bar{\psi}_0 = \bar{\psi}_0(\bar{Y})$ with region II determines, with the help of (3.45), $\bar{\psi}_0$ as the Blasius function, thus

$$\bar{\psi}_0 = f_B(\bar{Y}). \quad (3.48)$$

$O(\alpha^{\frac{1}{2}})$

$$\frac{\partial\bar{\psi}_0}{\partial\bar{Y}}\frac{\partial^2\bar{\psi}_1}{\partial\bar{X}\,\partial\bar{Y}} - \frac{\partial^2\bar{\psi}_0}{\partial\bar{Y}^2}\frac{\partial\bar{\psi}_1}{\partial\bar{X}} = 0,$$

or

$$\bar{\psi}^2_{0\bar{Y}}\frac{\partial}{\partial\bar{Y}}[\bar{\psi}_{1\bar{X}}/\bar{\psi}_{0\bar{Y}}] = 0. \quad (3.49)$$

Integrating we deduce from (3.49) that

$$\bar{\psi}_1 = F_1(\bar{X})f'_B(\bar{Y}) + \Gamma_1(\bar{Y}). \quad (3.50)$$

The matching condition (3.43) determines F_1 as $F_1(x) = -g(x)/\lambda$ and we note that this term makes a contribution $-\alpha^{\frac{1}{2}}g/\lambda$ to $\Phi(\bar{X},\alpha)$ in (3.44). The match with regions I, II and III is completed by setting $\Gamma_1 \equiv 0$.

$O(\alpha\log\alpha)$

As for $\bar{\psi}_1$ we calculate $\bar{\psi}_2$ as

$$\bar{\psi}_2 = F_2(\bar{X})f'_B(\bar{Y}) + \Gamma_2(\bar{Y}), \quad (3.51)$$

and matching with region III shows, using (3.43), that $F_2(x) \equiv 0$. Further, matching with region II gives

$$\Gamma_2(\bar{Y}) = -\tfrac{3}{2}\Lambda_1(\bar{Y}), \quad (3.52)$$

which is consistent with (3.44) and (3.45).

$O(\alpha)$

The terms $O(\alpha)$ give the following equation for $\bar{\psi}_3$,

$$\bar{\psi}_{3\bar{X}\bar{Y}}\bar{\psi}_{0\bar{Y}} - \bar{\psi}_{3\bar{X}}\bar{\psi}_{0\bar{Y}\bar{Y}} = \bar{X}^{-1} - (\bar{\psi}_{1\bar{Y}}\bar{\psi}_{1\bar{X}\bar{Y}} - \bar{\psi}_{1\bar{X}}\bar{\psi}_{1\bar{Y}\bar{Y}}), \quad (3.53)$$

which when rearranged, making use of (3.48), becomes

$$\frac{\partial}{\partial\bar{Y}}\left[\frac{\bar{\psi}_{3\bar{X}}}{f'_B}\right] = \frac{1}{\bar{X}f'^2_B} - \frac{g}{\lambda^2}\frac{dg}{d\bar{X}}\left\{\frac{f''^2_B - f'_Bf'''_B}{f'^2_B}\right\}, \quad (3.54)$$

from which we deduce

$$\bar{\psi}_3 = -\Lambda_1(\bar{Y})\log(-\bar{X}) + \frac{1}{2\lambda^2} g^2 f_B''(\bar{Y}) + f_B' F_3(\bar{X}) + \Gamma_3(\bar{Y}). \tag{3.55}$$

As $\bar{Y} \to 0$ in (3.55) we have

$$\bar{\psi}_3 \sim \frac{g^2}{2\lambda} - \frac{1}{\lambda}\log(-\bar{X}) + \lambda\bar{Y}F_3(\bar{X}) + \Gamma_3(\bar{Y}), \tag{3.56}$$

which, using (3.43), matches with region III if $F_3(\bar{X}) \equiv 0$ and $\Gamma_3 \sim (3/\lambda)\log\lambda\bar{Y}$ as $\bar{Y} \to 0$. Similarly, as $\bar{Y} \to \infty$ equation (3.55) with $F_3 \equiv 0$ gives

$$\bar{\psi}_3 \sim \bar{Y}\log(-\bar{X}) + \Gamma_3(\bar{Y}),$$

which, from (3.44), matches with region I if $\Gamma_3 \sim -2\bar{Y}\log\lambda$ as $\bar{Y} \to \infty$. The function $\Gamma_3(\bar{Y})$ is finally determined by matching with region II although, as we see from (3.45), this only enables us to express $\Gamma_3(\bar{Y})$ in terms of the undetermined $\Lambda_3(\bar{Y})$.

We do not continue our solution beyond this stage at which our analysis has revealed the detailed structure of the solutions in the various regions into which our flow field has been divided. We now consider some of the consequences of the foregoing analysis and in particular boundary layer separation.

4. Separation

It is now possible to consider the structure of the boundary layer near the trailing edge with reference to Stratford's method. The pressure distribution is one for which his method should be relevant as it consists of a long interval of slow variation followed by a rapid change, leading however to only a small increase of pressure. We have seen that the structure of the boundary layer in the critical region is as envisaged by Stratford physically, so that the basic ideas behind his method are correct. However using his explicit formula for separation (Stratford 1954, p. 7) we get for the value x_s of x at separation

$$(-x)_s = O\left[\alpha^{\frac{3}{4}}\left(\log\frac{1}{\alpha}\right)^{\frac{1}{2}}\right], \tag{4.1}$$

whereas it is clear from the preceding section that $(-x)_s = O(\alpha^{\frac{3}{2}})$. Stratford's method can be interpreted as a Pohlhausen approach to solving (2.13), (2.14) and it may be that the quantitative errors we have found can be removed by a different choice of profile. One tried here involved using exponential instead of algebraic terms and gave $(-x)_s = 1{\cdot}83\alpha^{\frac{3}{2}}$. A more satisfactory procedure is to use the series (3.22) and, after numerical integration of (3.24), the following series for $\tau = (\partial U/\partial Y)_{Y=0}$:

$$\tau = 1 - 1{\cdot}8575\chi - 0{\cdot}7314\chi^2 - 0{\cdot}9158\chi^3 - 1{\cdot}5124\chi^4 - \ldots, \tag{4.2}$$

where $\chi = (-X)^{-\frac{2}{3}}$, from which we deduce that

$$\tau^2 = 1 - 3{\cdot}7150\chi + 1{\cdot}9875\chi^2 + 0{\cdot}8855\chi^3 + 0{\cdot}9123\chi^4 \ldots \tag{4.3}$$

On truncating (4.2) after two, three, four or five terms the following estimates for χ_s are obtained

$$0{\cdot}338, \quad 0{\cdot}456, \quad 0{\cdot}428, \quad 0{\cdot}413; \tag{4.4}$$

similarly from (4.3) we obtain

$$0{\cdot}269, \quad 0{\cdot}326, \quad 0{\cdot}341, \quad 0{\cdot}347; \tag{4.5}$$

it is expected that the sequence (4.4) converges to the true value of χ_s from above whilst (4.5) converges from below. Further it is known that $\tau \sim (\chi_s-\chi)^{\frac{1}{4}}$ near $\chi = \chi_s$ while $\tau^2 =$ (regular function of $\chi) + O(\chi_s-\chi)^{\frac{3}{4}}$. Surmising that the rate of convergence of (4.5) is faster than that of (4.4) in the ratio $\frac{5}{4}:\frac{1}{2}$ we construct a new sequence by weighting (4.4), (4.5) with factors $\frac{2}{7}$, $\frac{5}{7}$ and adding. We get

$$0{\cdot}346, \quad 0{\cdot}361, \quad 0{\cdot}366, \quad 0{\cdot}366, \tag{4.6}$$

and infer that separation occurs at

$$\chi_s \doteqdot 0{\cdot}367 \quad \text{or} \quad (-x)_s \doteqdot 4{\cdot}50\alpha^{\frac{3}{4}}. \tag{4.7}$$

Thus neither Stratford's original method nor the simple modification tried above gives a satisfactory estimate for separation and there is a case for seeking a more reliable approximate method to deal with boundary layers involving rapid pressure variations.

5. The case $\alpha R^{\frac{1}{4}} \ll 1$

We have shown that, for the case of a wedge, separation of the boundary layer will take place at a distance $O(\alpha^{\frac{3}{4}})$ from the tip if $\alpha R^{\frac{1}{4}} \gg 1$. We have also suggested in § 2 that the largest trailing edge angle for which the flow will not separate is $O(R^{-\frac{1}{4}})$. We shall conclude this paper by showing, quite simply, that for the case $\alpha R^{\frac{1}{4}} \ll 1$ the flow is maintained up to the trailing edge.

To achieve this we use the results obtained by Stewartson (1968, 1969) for the flow in the neighbourhood of the trailing edge of a flat plate. Stewartson (1969) shows that at the trailing edge, between the region described by the Blasius boundary layer and that described by the Goldstein wake solution, is an intermediate region centred on the trailing edge and of scale ϵ^3 where $\epsilon = R^{-\frac{1}{8}}$ in which the pressure gradient induced in the inviscid flow field by the boundary layer can no longer be neglected. Let $(\bar{x}, \bar{y})$ be Cartesian co-ordinates on the scale ϵ^3 of the intermediate region with origin at the tip, $\bar{y}$ measured normal to the plate and $\bar{x}$ downstream. Stewartson then shows that the solution in the intermediate region has to be studied on three 'decks' whose thicknesses are $\bar{y} = O(1)$, $O(\epsilon)$, $O(\epsilon^2)$ respectively and that finally (Stewartson 1968) the solution is completed by solving the Navier–Stokes equations in a region of scale $O(\epsilon^3)$. An approximate solution is obtained for this final inner region. The relevance of Stewartson's work in the present problem is as follows. We note first that for the small angles ($\ll O(\epsilon^2)$) under consideration the boundary conditions on the wedge surface $\bar{y} = -\pi\alpha\bar{x}$, $\bar{x} < 0$ and on the dividing streamline $\bar{y} = 0$, $\bar{x} > 0$ may, to first order in α, be replaced by the same conditions on $\bar{y} = 0$ for all $\bar{x}$. Secondly, in this intermediate region $\psi' = O(\epsilon^3)$ and if $\epsilon^3\psi^* = \psi'$ then (2.1) shows that the boundary condition to be satisfied as $\bar{r} = (\bar{x}^2+\bar{y}^2)^{\frac{1}{2}} \to \infty$, $\bar{y} \neq 0$ is $\psi^* - \bar{r}\sin\theta \to 0$ to first order in α, exactly as for a flat plate. Thirdly, the work of § 3 (see for example (3.17)) shows that, for the small angles under consideration, as $\bar{x} \to -\infty$ for $0 \leqslant \bar{y} < \epsilon$ the solution approaches, to first order in α, the Blasius solution. Similarly as $\bar{x} \to \infty$ the solution will merge with the Goldstein wake solution. Consequently the problem as posed, to first order in α, is exactly as for the flow at the trailing edge of a flat plate and we can take over the work of Stewartson for both the intermediate and final inner regions. However, the flat plate solution cannot remain uniformly valid at the tip since ultimately the wedge-like nature must be apparent. On a scale which is sufficiently small the flow will be Stokesian in nature at the tip and for the flat plate the solution for the stream function is

$$\rho^{\frac{3}{2}}(\sin\tfrac{3}{2}\theta + \sin\tfrac{1}{2}\theta), \tag{5.1}$$

where ρ is distance measured from the trailing edge in this Stokes region. The corresponding solution for the wedge is

$$\rho^{\frac{3}{2}+\alpha}(\sin\tfrac{3}{2}\theta + \sin\tfrac{1}{2}\theta), \tag{5.2}$$

showing how the non-uniformity is accommodated.

The conclusion we draw from the above is that in the case of angles so small that $\alpha R^{\frac{1}{4}} \ll 1$ the flow will be maintained, without separation, up to the trailing edge.

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